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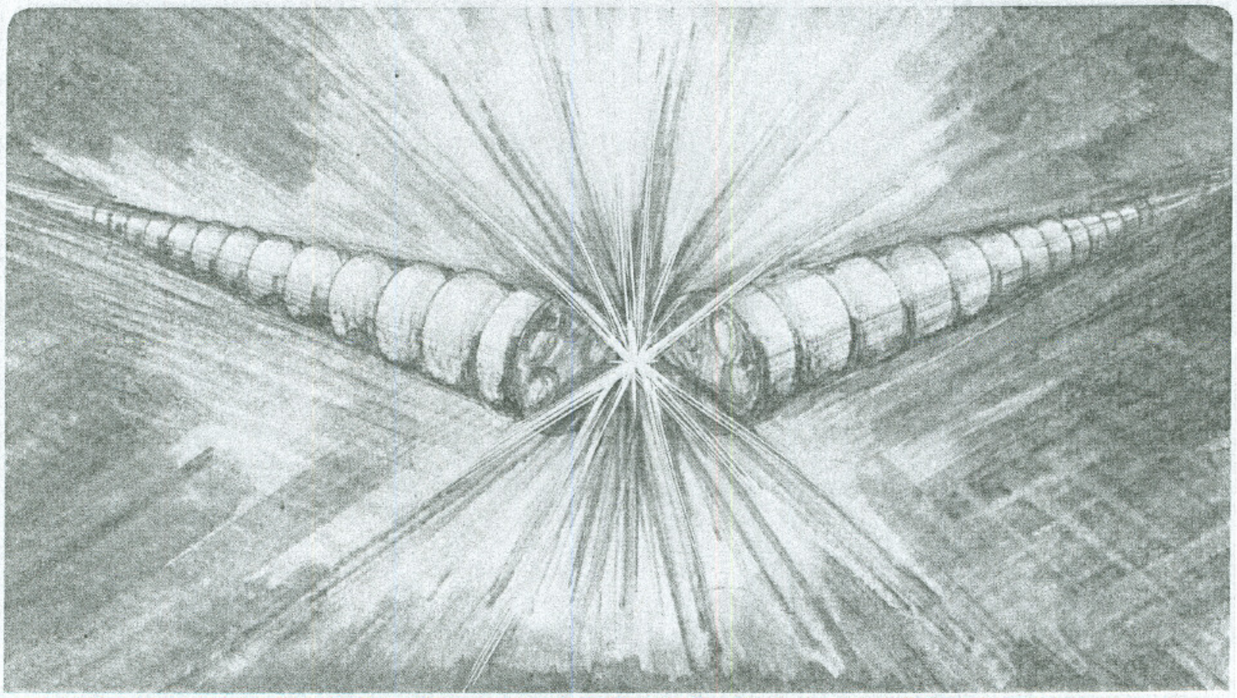

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SINGLE RESONANCE THEORY WITH MAPS*

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Abstract

We describe how one can perform a single resonance calculation using the single resonance map and the co-moving map. Again we emphasize that all the concepts of this paper have been implemented for the most complicated situations using the Differential Algebra Package of Berz. In theory, it is possible and desirable to do similar calculations on fitted maps (à la Warnock) with fitted canonical transformations, unfortunately the tools based on fitted maps are not yet as versatile as DA-based tools.

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Warning
The word **convergence** should no be taken in its purest mathematical sense throughout this paper. All the series produced in this paper are probably asymptotic. By convergence we mean to say that the asymptotic series is still well behaved as we increase the order of the perturbation to its chosen maximum value.

1. Introduction

In previous papers [1,2], we have argued that all the usual concept of single particle dynamics in a periodic Hamiltonian can be obtained from finite time maps. We have argued that both **thinking** and **computing** using finite time maps is a more efficient approach to accelerator physics than the usual Hamiltonian theory for two obvious reasons:

1) Circular rings are made of very discontinuous Hamiltonians in the time-like variable "s".

2) We are interested in the motion at a finite number of surface of section (or locations in the "s" variable).

Hence it is most efficient to define the ring mathematically as an ordered set of maps between the locations of interest.

From a computational standpoint, this view has been reinforced by the development of the DA-Package by Berz [3]. Given a tracking code, one can automatically get the exact coefficients of the Taylor series map around any orbit. This map can then be analyzed with Lie methods using a program also based on the DA-package of Berz. We have described this in detail in reference [4]. In particular, we hinted at the possibility of studying a "single-resonance" map. In this note, we will describe how one gets from a 2N phase space the N pseudo-invariants in a system dominated by one resonance. As usual, the claim is that all calculations can be done to arbitrary order with DA-based software and that the results can be compared with the actual tracking code which produced the map.

2. The single-resonance map

We start with the usual Taylor series expansion:

$$z^f{}_i = T^1{}_{ij} z_j + T^2{}_{ij_1j_2} z_{j_1} z_{j_2} + \ldots + T^{no-1}{}_{ij} z_{j_1} \ldots z_{j_{no-1}} + \ldots$$
$$\mathbf{z} = (\mathbf{q},\mathbf{p}) \tag{1}$$

We know that a symplectic map **M** acting on functions of phase space can be found from equation (1). For example, it is possible to write **M** as a factored product (Dragt-Finn) [5]:

$$\mathbf{M} = \exp(:f_2:)\ \exp(:f_3:)\ \ldots.\exp(:f_{no}:) \tag{2}$$

such that

$$\mathbf{z^f} = \mathbf{M\ z} = T^1{}_{ij} z_j + T^2{}_{ij_1j_2} z_{j_1} z_{j_2} + \ldots + T^{no-1}{}_{ij} z_{j_1} \ldots z_{j_{no-1}} + \ldots.O(||\mathbf{z}||^{no}) \tag{3}$$

This factorization is not really useful except for the fact that it assumes no special properties of the map **M**. Here however, we will assume that **M** is dominated by one resonance and that between the location of this resonance in action and the origin there appear to be no resonances (as depicted on figure 1). Let us call this region of phase space B . Within B, one can attempt the following transformation:

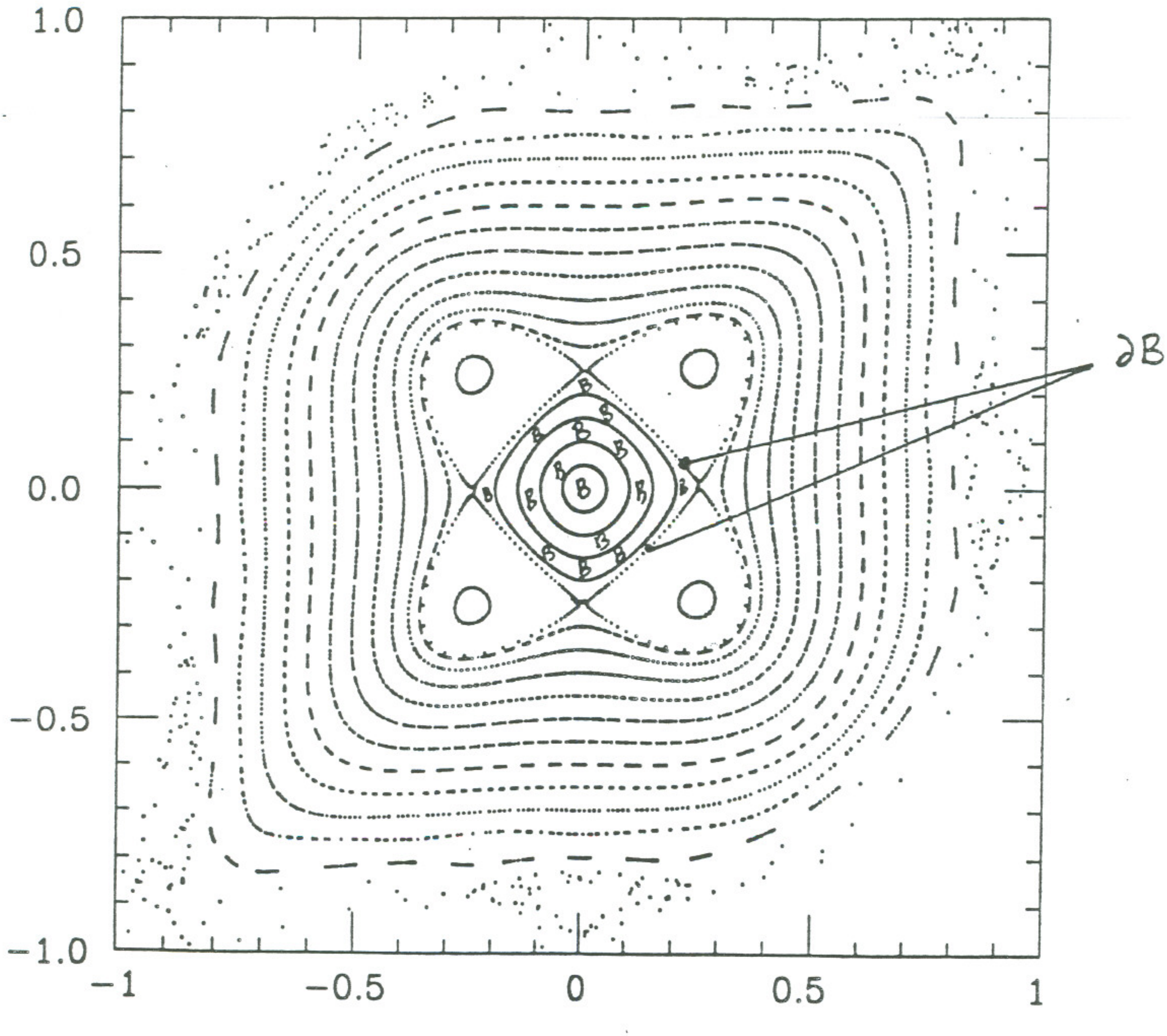
$$\mathbf{M} = \mathbf{A}^{-1}\mathbf{R}\mathbf{A} + \ldots O(||\mathbf{z}||^{no}) \text{ in the Taylor series}$$

where $\quad \mathbf{R} = \exp(:-\mu\cdot\mathbf{J} + D(\mathbf{J}):)$

and $\quad \mathbf{A} = \exp(:F_{no}:)\ \ldots\exp(:F_3:)\ \mathbf{A_2}\ldots. \quad (4)$

The computation of **A** from a DA-map has been described in detail in reference 4. It should be said that **A** is bound to diverge somewhere inside the domain B. If we restrict ourselves to cases where **A** diverges at ∂B, we can introduce the transformation $\mathbf{A_r}$ such that:

$$\mathbf{M} = \mathbf{A_r}^{-1}\mathbf{N}\mathbf{A_r}$$



1. Location of B, ∂B illustrated on a one dimensional map.

where

$$\mathbf{N} = \exp(:-\mu\cdot\mathbf{J}:)\ \exp(:T_3:)\ \ldots.\exp(:T_{no}:)$$

$$T_i = D_i(\mathbf{J}) + \sum_{\mathbf{j}-\mathbf{k}=\lambda\mathbf{m}} A^i_{\mathbf{j},\mathbf{k}}|\mathbf{j},\mathbf{k}> \quad ; \qquad \lambda\in Z \tag{5}$$

$$|\mathbf{j},\mathbf{k}> = (h^+_1)^{j_1}(h^-_1)^{k_1} \ldots (h^+_N)^{j_N}(h^-_N)^{k_N},$$
$$\text{with} \quad h^{\pm}_j = q_j \pm i\, p_j \quad . \tag{6}$$

The vector $\mathbf{m}$ is the defining vector of the resonance:

$$\mathbf{m}\cdot\mu(\mathbf{J}) = p \qquad\qquad p = \text{ integer} \tag{7}$$

Notice that the map $\mathbf{N}$ retains all the resonant terms.To the extent that the divergences in equation (4) are due to the small denominators in $\mathbf{A}$ which are produced by the $|\mathbf{j},\mathbf{k}>$ terms with $\mathbf{j}-\mathbf{k}=\lambda\mathbf{m}$, the map $\mathbf{N}$ is therefore well behaved.

In addition the single-resonance map $\mathbf{N}$ is "closed". By this we simply mean that any nonlinear manipulation on $\mathbf{N}$ will always produce a single-resonance map. This is true because the set

$$S_{\mathbf{m}} = \{\ |\mathbf{j},\mathbf{k}> \mid \mathbf{j}-\mathbf{k}=\lambda\mathbf{m}\ ;\ \lambda\in Z\} \tag{8}$$

is closed under Poisson bracket.

3. The orthogonal invariants of N

The particular form of $\mathbf{N}$ allows for the immediate computation of N-1 invariants. Consider an arbitrary linear combination of the actions J_i:

$$I_{\mathbf{a}} = \mathbf{a}\cdot\mathbf{J} \tag{9}$$

Let us compute the Poisson bracket of $I_{\mathbf{a}}$ with $|\mathbf{j},\mathbf{k}>$:

$$[\mathbf{a}\cdot\mathbf{J},|\mathbf{j},\mathbf{k}>] = \{i(\mathbf{j}-\mathbf{k})\cdot\mathbf{a}\}\ |\mathbf{j},\mathbf{k}> = i\lambda\mathbf{m}\cdot\mathbf{a}\ |\mathbf{j},\mathbf{k}> \qquad i=(-1)^{1/2} \quad (10)$$

From equation (10) and the equation for **N**, we conclude that if $\mathbf{m}\cdot\mathbf{a}=0$ then $I_{\mathbf{a}}$ is an invariant of the map **N**. Since there are N-1 vectors orthogonal to **m**, **N** has N-1 invariants of the type $I_{\mathbf{a}}$. The corresponding invariants $K_{\mathbf{a}}$ of the map **M** are given by the usual prescription:

$$K_{\mathbf{a}} = \mathbf{A}_{\mathbf{r}}^{-1}\ I_{\mathbf{a}} \qquad (11)$$

4. The co-moving map $\mathbf{N_c}$

As it is done in Hamiltonian theory, we can now introduce a co-moving phase ϕ_c :

$$\phi_c = \mathbf{m}\cdot\vec{\phi} \qquad (12)$$

In fact using the new N-1 invariants, we introduce a new set of angles and actions:

$$\Phi_{i+1} = \mathbf{a}^i\cdot\vec{\phi}\ ;\ K_{i+1} = \mathbf{a}^i\cdot\mathbf{J} \quad \text{with} \quad i=1,N-1\ ,\mathbf{m}\cdot\mathbf{a}^i = 0,\ \text{and}\ ||\mathbf{a}^i||=1$$

$$\Phi_1 = \phi_c = \mathbf{m}\cdot\vec{\phi} \qquad K_1 = \mathbf{m}\cdot\mathbf{J}/||\mathbf{m}||^2 \qquad (13)$$

Using the co-moving action K_1 ,we construct the co-moving map $\mathbf{N_c}$:

$$\begin{aligned}\mathbf{N_c} &= \exp(:p2\pi\, K_1:)\ \mathbf{N}\\ &= \exp(:p2\pi\, K_1-\mu\cdot\mathbf{J}:)\ \exp(:T_3:)\ ....\exp(:T_{no}:) \qquad (14)\end{aligned}$$

The map has $\mathbf{N_c}$ has two fundamental properties:

i) It has the same invariants in B as the map **N**.
ii) The map $\mathbf{N_c}$ can be written with a single exponent which is still invariant in a region extending beyond the domain B.

Proof of i)

We must evaluate the canonical transformation **B** such that:

$$\mathbf{N}= \mathbf{B}^{-1}\mathbf{RB} \text{ where } \mathbf{R} = \exp(:-\mu\cdot\mathbf{J}+ D(\mathbf{J}):) \quad (15)$$

Lemma

$$\exp(:p2\pi\, K_1:)\ |\mathbf{j},\mathbf{k}> = |\mathbf{j},\mathbf{k}> \text{ if } |\mathbf{j},\mathbf{k}>\in S_{\mathbf{m}} \qquad (16)$$

Using equation (10) and the definition of K_1:

$$\exp(:p2\pi\, K_1:)\ |\mathbf{j},\mathbf{k}> = \exp(\ i\ p2\pi\ \mathbf{m}\cdot(\mathbf{j}-\mathbf{k})/||\mathbf{m}||^2\ )\ |\mathbf{j},\mathbf{k}> \qquad (17)$$

but since $|\mathbf{j},\mathbf{k}>\in S_{\mathbf{m}}$ there exists an integer λ such that $\mathbf{j}-\mathbf{k}=\lambda\mathbf{m}$. Hence equation (17) simplifies to

$$\exp(\ i\ p2\pi\ \mathbf{m}\cdot(\mathbf{j}-\mathbf{k})/||\mathbf{m}||^2\ )\ |\mathbf{j},\mathbf{k}> = \exp(\ i\ p2\pi\ \lambda)\ |\mathbf{j},\mathbf{k}> = |\mathbf{j},\mathbf{k}> \quad (18)$$

This proves the lemma.

Going back to equation (15) and multiplying it by $\exp(:p2\pi\, K_1:)$, we get :

$$\exp(:p2\pi\, K_1:)\ \mathbf{N} = \exp(:p2\pi\, K_1:)\mathbf{B}^{-1}\mathbf{RB}$$

$$\Rightarrow \quad \mathbf{N_c} = \exp(:p2\pi\, K_1:)\ \mathbf{B}^{-1}\mathbf{RB}$$

$$\Rightarrow \quad \mathbf{N_c}= \exp(:p2\pi\, K_1:)\ \mathbf{B}^{-1}\exp(:-p2\pi\, K_1:)\ \exp(:p2\pi\, K_1:)\ \mathbf{RB} \quad (19)$$

However, using closure of $S_{\mathbf{m}}$, we conclude that the map $\mathbf{B}^{-1}$ contains Lie exponents in $S_{\mathbf{m}}$ only, hence, by application of the lemma, we obtain the final result:

$$\mathbf{N}= \mathbf{B}^{-1}\mathbf{R_c}\mathbf{B}$$

where $\mathbf{R_c} = \exp(:p2\pi\, K_1:)\ \mathbf{R} = \exp(:p2\pi\, K_1-\mu\cdot\mathbf{J}+ D(\mathbf{J}):) \qquad (20)$

The N pseudo-invariants around the origin are given by

$$I_i = \mathbf{B}^{-1} K_i \tag{21}$$

For i=2,N, we have simply $I_i = K_i$. For i=1, the invariant diverges again at the boundary ∂B.

Heuristic discussion of ii)

Ignoring the divergence of I_1, let us write $\mathbf{N_c}$ as a single exponent:

$$\begin{aligned}\mathbf{N_c} = \mathbf{B^{-1} R_c B} &= \exp(\mathbf{B^{-1}}\mathbf{:}p2\pi\, K_1 - \mu\cdot\mathbf{J} + D(\mathbf{J})\mathbf{:B}) \\ &= \exp(\mathbf{:B^{-1}}\{p2\pi\, K_1 - \mu\cdot\mathbf{J} + D(\mathbf{J})\}\mathbf{:}) \\ &= \exp(:-H(\mathbf{J};\Phi):) \end{aligned} \tag{22}$$

The pseudo-Hamiltonian $H(\mathbf{J};\Phi)$ is in the set $S_{\mathbf{m}}$. To see why $H(\mathbf{J};\Phi)$ might be a good invariant beyond ∂B, we need to express the map $\mathbf{N_c}$ of equation (14) using the co-moving actions and angles:

$$p2\pi\, K_1 - \mu\cdot\mathbf{J} = (p2\pi - \mathbf{m}\cdot\mu)\, K_1 - \sum_{i=2,N} \mu\cdot\mathbf{a^i}\, K_i \tag{22}$$

Presumably, the zeroth order tune μ is not very far from resonance condition:

$$(p2\pi - \mathbf{m}\cdot\mu) = \varepsilon << \mathbf{m}\cdot\mu \tag{23}$$

Hence, the operator $\mathbf{N_c}$ of line (14) can be rewritten as

$$\mathbf{N_c} = \exp(:\, \varepsilon K_1 - \sum_{i=2,N} \mu\cdot\mathbf{a^i}\, K_i:)\, \exp(:T_3:) \,....\exp(:T_{no}:) \tag{24}$$

If we try to rewrite (24) as a single exponent using the Campbell-Baker-Hausdorff (CBH) formula, its convergence will be controlled by the parameter ε. In the original map $\mathbf{N}$, the parameter in front of K_1 was $-\mathbf{m}\cdot\mu$; a parameter assumed to be in the range of $p2\pi$ and therefore restricting the convergence of the CBH formula within B.

However, there is a more geometrical reason to expect $H(\mathbf{J};\Phi)$ to be a good invariant beyond ∂B. Geometrically, the factor $\exp(:p2\pi K_1:)$, in equation (14), takes a point propelled forward by **N** and brings it back in the neighborhood of the initial condition. At the resonance, the angles ϕ do not occupy the entire open set $(0,2\pi)^N$, but are congregated into clumps or islands. The map $\exp(:p2\pi K_1:)$ prevents a phase space point from leaving a given island. Hence the tune of the map $\mathbf{N_c}$ drifts from nearly zero at the origin to zero at the island and away from zero beyond the island. This type of motion is topologically compatible with the motion produced by a time-independent Hamiltonian. Therefore, the small denominator divergences in **B** are cancelled by the terms proportional to the small numerator ε in equation (22).

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